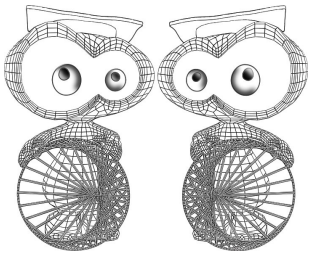

Cesenatico 2010

# 11th ITALIAN TEAM COMPETITION

Final – 8 Maggio 2010

**General rules**

- Each problem requires an answer which is an integer number between 0000 e 9999. The answer must be marked on a card.
- If the answer is not an integer, unless otherwise stated, mark its rounding to the nearest integer.
- If the answer is not a negative integer, or if the problem has no solution, mark 0000.
- If the answer is an integer greater than 9999, mark the last four digits.
- it may be useful to know the following approximations:

$$\sqrt{2} = 1.4142 \qquad \sqrt{3} = 1.7321 \qquad \sqrt{5} = 2.2361 \qquad \sqrt{7} = 2.6458 \qquad \pi = 3.1416.$$

**Important deadlines**

- **10 minutes from the start**: deadline to choose the special problem (after that, the first problem will be the special one).
- **30 minutes from the start**: deadline to ask questions about the text.
- **120 minutes from the start**: end of the competition.

## 10 Easy Pieces

### 1. The Scumm Bar™

At the dawn of 8th May 1726, Jack Arrow, Bourbakossa, and Elizabeth Summ are getting drunk at a bar when they are approached by a naive-looking young man who says that he wants to become a π-rate, and that he has been told that he must look for three Important-Looking Π-rates. The three friends ask him, "How many years are there between now and 2010 that are multiples of the answer to this question?"

### 2. Battle of the Fleets

In the epic battle between the π-rates and the East India Trading Company, whose tale will be told for generations to come, the $n$ ships led by Jack Arrow defeated the $m$ ships of Lord Bracket's fleet, where $n$ and $m$ are the solutions to the equation $x^2 - 126x + 3293 = 0$. Knowing that each of Arrow's $n$ ships was crewed by $n$ π-rates, and each of the Company's ships was crewed by $m$ soldiers, how many people participated in the battle?

### 3. Noble π-rate and aesthete

Many are surprised to discover that the bloody Bourbakossa is a refined aesthete. He wants to tidy his wine cellar after Jack's visit. He owns 28 identical bottles of rum, and he wants to arrange them on 7 shelves in such a way that:

- there are a different number of bottles on every shelf;
- for every $n = 1, 2, \ldots, 7$ the total number of bottles on the first $n$ shelves is *not* a multiple of 3.

In how many ways is it possible to do this so that no shelf is empty?

### 4. The pendulum to safety

Some π-rates have been captured by cannibals and imprisoned in a cage tied to a long rope hanging in the middle of a canyon. To escape, they must swing the rope until it reaches the wall of the canyon. Elizabeth calculates that the number of necessary oscillations is the sum of all positive integers which are equal to the square of the sum of their own digits. How many oscillations will they need, if her computations are correct?

### 5. Musical piracy

All π-rates love music, but the true purists prefer not to pay for it, so they resort to "browsing" until they find a chance to "unload" it illegally from captured ships. Jack is looking for a cargo of Π-rate Minstrels, and to intercept their ship, he must find all four-digit numbers such that:

- both the sum of the first and third digits and the sum of the second and fourth digits are equal to 9;
- there are no consecutive odd digits;
- the first digit is less than the second and fourth digits.

The sum of all numbers that satisfy these conditions will give the coordinates of the ship. What is this sum?

### 6. The morning run

Before becoming a feared π-rate, Will Turing was a weaponsmith and amateur swordsman. He kept himself fit by jogging every day from his shop down to the docks, 5 km away. Will tells Elizabeth that on flat ground his average speed was 20 km/h; downhill, he could reach 30 km/h, while uphill he could only manage 15 km/h. Will challenges Elizabeth to guess how much time, at most, he ran each day. What is the answer?

### 7. The crazy compass

Jack Arrow's compass almost never points north. At midnight on a certain day, the compass points north, and then begins to move following this algorithm: every 5 minutes, the needle suddenly moves forward by 90 degrees, then immediately rotates back by a number of degrees equal to 6 times the number of minutes since the midnight when we started counting. For how many minutes a week does the compass point north?

### 8. The Sargasso Tavern

The π-rate crew of the True Pearl are eating at the Sargasso Tavern. They want to move 5 square tables together (so that some of their sides touch) to form a single table for 12 people. (Each table seats one person on each side.) How many ways are there to arrange the tables that satisfies the conditions? (Solutions which differ by rotation are different, while solutions which are symmetrical should not be considered distinct.)

### 9. At the World's Perimeter

Bourbakossa has finally managed to steal the Map of the World's Perimeter from the π-rate Sao Feng, and he's studying it aboard his vessel. Moving and rotating the various pieces of the map, Bourbakossa forms a regular hexagon. Then, he adds as many regular pentagons outside of it as possible, so that each shares a side with the previous figure, with no overlap. Then, he repeats the process, adding all possible squares, and after that, all possible equilateral triangles, always so that there is no overlap between the figures. Let $a$ be the the number of regular polygons that make up the resulting figure, and $b$ the measure in degrees of the smallest non-zero angle formed by any sides with a common vertex. What is the product $ab$?

### 10. For one hand more

Because of frequent duels, π-rates are very likely to lose a hand in combat. For this reason, π-rates count in base 5 instead of base 10. Ragetti, the ship's boy with a wooden eye is playing the following game: he writes down on a piece of paper all numbers in base 5 that contain every digit from 0 to 4 exactly once (0 can be the initial digit). Lastly, he calculates the average of the numbers he wrote down. What number does he obtain, in base 10?

## 14 Not So Easy Pieces

### 11. Who seeks Davy Jensen's favours?

The cursed π-rate Davy Jensen has a bag which contains 90 white chips, numbered from 1 to 90, and 1 chip that's blood red. Whoever seeks his favours, in order to find how many years he will benefit, must play the following game: he draws chips one at a time at random until he draws the red one. The player will have favours for as many years as the number of white chips he has drawn whose number is a multiple of 7. "Bootstrap Bill" Turing, watching from a corner, wonders what the average numbers of years won is. Jack doesn't understand what his fellow π-rate means. Bill's son Will, who has an education, explains that there are many ways to find the average number of years won (all of which produce the same result), but the most common is to find the sum of all the numbers $k \cdot p_k$ where, for each natural number $0 \leq k \leq 12$, $p_k$ is the probability that the years won will be $k$. What is the average number of years won?
[As the answer, provide the product of the numerator and the denominator of the fraction in its simplest form.]

### 12. The mouth of the Cramer

No one knows exactly what shape the mouth of the Cramer is, or, at least, no one has ever lived to tell the tale. One legend, however, tells of a certain triangular shape... Let $PQR$ be a triangle with sia $QR < PR < PQ$, and let $S$ be the intersection of the bisector of the exterior angle in $P$ and the line $QR$. Let $T$ be the intersection of the bisector of the exterior angle in $R$ and the line $PQ$. Knowing that $SP = PR = RT$, find the measure of the angle $\widehat{QRP}$ in minutes of arc. (A minute of arc is $1/60$ of a degree.)

### 13. Ante up

Will Turing never lived down his defeat in the duel with Jack Arrow when they first met. Every year, for 40 years, they meet and duel; each bet the same amount of money, and at the end the winner takes all. The first year, each one bets $a_1 = 2010$ doubloons, then the following years they bet the amount of money given by the formula $a_{n+1} = 2a_n + 2n - 1$ where $n = 1, .., 39$. At the end of the 40 duels, Will has won the most doubloons, but Jack is satisfied because he has lost the least possible amount. How many doubloons did Will win?

### 14. The anti-Cramer talisman

Jack is still hunted by the Cramer, which can sense his presence anywhere, and follows him without rest. The voodoo lady Thia Djeome, using her Geomystical power, helps him build a talisman to keep the monster at bay. She explains that he will need to make a sphere of clay, and on its surface he must trace two mystical circumferences. These circumferences must have diameters of 50 and 36 cm, they must lie on orthogonal planes, and they must intersect in two points 14 cm apart? How long should the diameter of the sphere be, in *millimeters*?

### 15. The captain's organ

As all melancholy captains, Davy Jensen plays the organ. On the Lined Dutchman he owns a very particular one, which has 9999 keys, numbered from 1 to 9999. The keys whose numbers can be written as $[2x] + [4x] + [8x] + [12x]$ for some real x are white. All the other keys are black. How many of the keys are white? (The symbol $[\cdot]$ denotes the floor function. Given a real number $y$, $[y]$ is the greatest integer which is less than or equal to $y$.)

### 16. The delirium of Jack Arrow

Jack was not able to escape the Cramer, and the monster ate him and the True Pearl. Now, imprisoned beyond the World's Perimeter, he is plagued by hallucinations. From the mast of the True Pearl, he looks down and sees 144 crabs standing on the vertices of a regular polygon with 144 sides. Still hallucinating, Jack wonders how many non-congruent triangles whose vertices are vertices of the polygon. How many are they? (Recall that triangles which are symmetrical to each other with respect to a line are congruent.)

### 17. Davy Jensen's Locker

Jack and his crew are prisoners in Davy Jensen's Locker, the mystical waters beyond the World's Perimeter. At sunset, Captain Arrow figures out the way to return to the world of the living: he must turn the ship upside down, and he must find three positive integers, such that $a + b + c = 2010$, which can be turned upside down as well, so that $1/a + 1/b + 1/c = 1/58$. What is the least common multiple of the three integers?

### 18. Jack Arrow's compass

Jack Arrow's magical compass was made by Thia Djeome, with rigorous Geomystical criteria. The compass has the shape of a hexagon, obtained as follows: we begin with a right triangle whose catheti have integral lengths in mm; we draw three external squares on its sides; we then draw segments connecting the vertices of the squares that do not lie on the triangle, two by two, to obtain a convex hexagon. Knowing that the surface of the hexagon measures $1922\,\text{mm}^2$, find the area of the right triangle we started with.

### 19. Cutler Bracket's trouble

The Brethren Court of noble $\pi$-rates convene at a secret location known as Shipwreck Cove. To relay the coordinates to new affiliates, they use coded messages that only true $\pi$-rates can decipher. Lord Bracket has intercepted one, and for days he has racked his brain with no success. The message reads, "The coordinates are three positive integers $a, b, c$, such that none of the three is a multiple of another,

$$\text{lcm}(a,b) \cdot \text{lcm}(b,c) \cdot \text{lcm}(a,c) = a \cdot b \cdot c \cdot \gcd(a,b,c)$$

and $a + b + c$ has the least possible value. What is $a + b + c$?

### 20. The dog with the keys

After the last treachery, Jack Arrow is locked in a cell in the Lined Dutchman. Without fail, the dog Spiffy walks in holding the keys, and Jack attempts to attract him with no success. "Bootstrap Bill" Turing emerges from the darkness of the cell, and explains, "The dog only answers to those who know the magic numbers $a$ and $b$ and state their product. I'll give you a few hints: $a$ is a positive integer of two distinct digits, $b$ is obtained from $a$ by switching the digits, and $a^2 - b^2$ is a square number which has the least possible value it can have." What number must Jack say to attract Spiffy?

### 21. The two maps

There are two maps that lead to the Isla de Mono. On the first map, which is square-shaped and whose sides measure 1726 miles, the 4 major ports of the East India Trading Company each lie on one side. By some strange coincidence, on the second map, which is also square and whose sides measure 1250 miles, these four ports lie on the vertices. Both maps are divided into squares whose sides measure 1 mile by blood-coloured lines parallel to the edges of the maps. Jack knows that, on both the first and the second map, the Isla de Mono lies on the intersection of two such blood-coloured lines (this can include the edges of the map). He does not know, however, which lines. How many points are there on which the island could lie?

### 22. The Secret of Monkey Island™

Every experienced π-rate has heard of the fabulous treasure hidden on the lost Isla de Mono; as the legend goes, the treasure was buried in a point on the island which is at least 5 m above sea level. Thia Djeome owns what may be the only map of the island, from which it can be deduced that:

- the island has a shape of a pentagon in which a circumference can be inscribed;
- the depth of the sea around the island is equal to $\sqrt{\pi}$ times the least distance from the coast measured on the map, while the altitude of every point on the surface of the island is equal to $\sqrt{\pi}/2$ times the least distance of the point from the water, measured on the map;
- the area enclosed by the points that are 1 metre below sea level measures $12343\,\mathrm{m}^2$, while the area enclosed by the points that are 3 metres below sea level measures $12835\,\mathrm{m}^2$.

What is the area, in $\mathrm{m}^2$, of the part of the island in which the treasure could be found, according to the legend?

### 23. A trap for fools

Left without a ship in Tortuga, Master Gibbs earns a living swindling passersby. The victim draws a number from 1 to 10000 at random from a bag. Then he begins, iteratively, to substitute the number with the product of its digits, and he numbers that are not less than 9 indefinitely. What are the odds of winning?
[As the answer, provide the product of the numerator and the denominator of the fraction in its simplest form.]

### 24. Shipwreck Cove

The Brethren Court's impregnable stronghold is built entirely with wood obtained from shipwrecks. It is in the shape of four identical cones, lying on flat ground so that they share vertices. Their lateral surfaces touch the ground on 4 equal perpendicular segments, forming a cross, and the lateral surfaces of two adjacent cones are tangent. Write the first four decimal digits of the ratio of the height to the base radius of the cones.